DYNAMICS OF SYMMETRIC SSVI SMILES AND IMPLIED VOLATILITY BUBBLES

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Abstract. We develop a dynamic version of the SSVI parameterisation for the total implied variance, ensuring that European vanilla option prices are martingales, hence preventing the occurrence of arbitrage, both static and dynamic. Insisting on the constraint that the total implied variance needs to be null at the maturity of the option, we show that no model–in our setting–allows for such behaviour. This naturally gives rise to the concept of implied volatility bubbles, whereby trading in an arbitrage-free way is only possible during part of the life of the contract, but not all the way until expiry.

1. Introduction

Implied volatility is at the very core of financial markets, and provides a unifying and homogeneous quoting mechanism for option prices. The literature abounds in stochastic models for stock prices that generate implied volatility smiles–with various degrees of practical success. Among those, the Heston model [23] in equity and the SABR model [20] in interest rates–together with their ad hoc and in-house improvements– have been of particular importance. Despite this success, these stochastic models do not enjoy the simplicity of closed-form expressions, and advanced numerical techniques are needed to implement them. One way to bypass this, for example, has been to consider approximations of option prices–and the corresponding implied volatilities–in asymptotic regimes; thorough reviews of the latter are available in [11, 12, 22, 30]. A different approach, pioneered by Gatheral [14], consists in specifying a direct parameterisation of the implied volatility, which has the clear advantage of speeding up computation and calibration times. The original Stochastic Volatility Inspired (SVI) formulation, devised while its inventor was at Merrill Lynch, has proved extremely efficient in fitting volatility smiles on equity markets. That said, it was only devised as a maturity slice interpolator and extrapolator, and different sets of parameters were needed in order to fit a whole surface (in strike and maturity). Gatheral and Jacquier [17] extended it to a whole surface, devising tractable sufficient conditions ensuring absence of arbitrage. The design of calibration algorithms is then easy, and this SSVI formulation has been adopted widely in the financial industry, and has since been extended [21, 7] to a version with maturity-dependent correlation.

SSVI does not depend on a model in the sense that it directly tackles option prices (equivalently, implied volatilities), without following the usual route of specifying a model for the evolution of the underlying. It is furthermore, by nature, fully static, as its inputs are market option prices at a given point in calendar time, with only strike and expiry allowed to vary. Gatheral and Jacquier [16] showed that, as the maturity increases, the SVI parameterisation was in fact the true limit of the Heston smile. A very natural question is therefore whether there exists a dynamic model such that at each calendar time, the option smiles in this model are given by SVI–or its arbitrage-free SSVI extension.

We propose here an answer to this question, albeit through a slightly different lens, as we investigate whether one can impose stochastic dynamics on the implied volatility, ensuring that arbitrage cannot occur over time. We work in a simplified and minimal setup, in a perfect market with no interest rates, in continuous time, and consider European Call/Put options with a fixed expiry, so we will restrict ourselves to the dynamic of a fixed smile; we will also assume that the underlying process does not distribute coupons nor dividends and does not default. Motivated by the discussion above, we will assume that the total implied variance has an SSVI shape at all (calendar) times before maturity, but is allowed to move stochastically, with the condition that both the underlying and option prices should be martingales.

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This is not the first suggested solution to this problem, and several authors have attempted to propose joint dynamics for the underlying stock price and the implied volatility. Motivated by empirical evidence that the implied volatility moves over time, the usual approach is to specify a stochastic Itô diffusion for the total implied variance, as in [4, 6, 19, 25, 29, 31, 32, 35]. However, to quote [8], *'the problem with market models is the extremely awkward set of conditions required for absence of arbitrage'*. An important step was made by Schweizer and Wissel [36, 37], who derived general conditions ensuring existence of such market models. Even if the resulting conditions are not easily tractable for modelling purposes, the work by Schweizer and Wissel is the first positive result and can be considered as an achievement from this point of view. The only other positive result (for continuous processes) we are aware of can be found in Babbar's PhD thesis [2], which, building on [31], developed stochastic models for the joint stock price and the total implied variance (which she calls the operational time), for a fixed strike, relying on comparison theorems for Bessel processes. One fundamental catch, though, is that the implied volatility may hit zero strictly before the maturity of the option, making the model degenerate. We shall revisit this degeneracy somehow, giving it some financial meaning.

The implied Remaining Variance framework [4, 5] shares a common point of view with our approach. There, the shape of the dynamic of the total implied variance is prescribed, whereas we derive it from the shape of the smile (SSVI in our case). Also we identify the terminal condition on the total variance at maturity as a key property, and prove that there are no processes satisfying this condition in our case.

Indeed our main result is disappointingly negative: starting from an uncorrelated SSVI smile at all times, we show that no Itô process, beyond the Black-Scholes model with time-dependent volatility, exists such that the option prices are martingales. The by-products of this result, however, are interesting and informative. We obtain explicitly joint dynamics for the underlying and the option prices such that, locally in time (that is until some time before the true maturity of the option), the underlying price is a martingale, and so are all the vanilla option prices, despite the fact that the option prices are not given by the expectation of the final payoffs under an equivalent martingale measure. This implies that until this horizon, it is not possible to synthesise an arbitrage. Yet, this cannot last until the maturity of the option, and the market will then change regime. This naturally gives rise to the new concept of implied volatility bubbles. We believe this intermediate regime (in the sense that it lies between the traditional arbitrage-free situation with a given specified dynamic until maturity and a regime with instantaneous arbitrages) is of interest, and may correspond to real world situations.

We introduce precisely the SSVI parameterisation in Section 2, and recall the notions of absence of arbitrage for a given implied volatility surface. In Section 3, we introduce a new stochastic model describing the dynamics of the implied volatility surface, and extend the static arbitrage concept to a dynamic version. We show there that unfortunately there cannot be any Itô process solution in our setting. However, this leads us to introduce implied volatility bubbles in Section 4, which we study in detail in the SSVI case.

2. Static arbitrage-free volatility surfaces

We recall in this section the key ingredients of volatility surface parameterisation as well as the different concepts of no (static) arbitrage in this setting. This will serve as the basis of our analysis, and allows us to define properly the notion of dynamic arbitrage, and its relation with martingale concepts. In order to set the notations, recall that, in the Black-Scholes model [3] with volatility $\sigma > 0$, the price of a Call option with strike $K > 0$ and maturity $T > 0$ is given at time $t \in [0,T]$ by

$$\mathrm{C}^{\mathrm{BS}}(S_t, K, T, \sigma) = \mathbb{E}\left[(S_T - K)_+ | \mathcal{F}_t\right] = S_t \mathrm{BS}\left(\log\left(\frac{K}{S_t}\right), \sigma\sqrt{T-t}\right), \tag{1}$$

for any $t \in [0,T]$, where the function $\mathrm{BS} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\mathrm{BS}(k,v) := \begin{cases} \mathcal{N}\left(d_+(k,v)\right) - \mathrm{e}^k \mathcal{N}\left(d_-(k,v)\right), & \text{if } v > 0, \\ \left(1 - \mathrm{e}^k\right)_+, & \text{if } v = 0, \end{cases} \tag{2}$$

with $\mathcal{N}$ being the Gaussian cumulative distribution function, and $d_\pm(u,v) := \frac{-u}{v} \pm \frac{v}{2}$. In order to normalise units and to be consistent, practitioners generally do not work with option prices directly, but rather with the Black-Scholes implied volatility map $\sigma_t : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ defined through the implicit relationship $C_t^{\mathrm{obs}}(K,T) = S_t \mathrm{BS}\left(k, \sigma_t(k,T)\sqrt{T-t}\right)$, where $C_t^{\mathrm{obs}}(K,T)$ denotes the observed option price at time t for a given strike and maturity, and $k := \log(K/S_t)$ is the log-moneyness. There exist different conventions to write the implied volatility; for reasons that will become apparent later, we choose to write it as a function of k. For convenience, we shall in fact work in terms of the total variance $\omega_t(k,T) := (T-t)\sigma_t^2(k,T)$, so that the Call price formula (1) can be rewritten, for any $t \in [0,T]$, as

$$C_t^{\text{obs}}(K,T) = S_t \text{BS}\left(k, \sqrt{\omega_t(k,T)}\right).$$

2.1. **Static arbitrage.** Following [17], we recall the notion of static arbitrage. By static, we mean that the running time t is fixed, and we are only interested in trading occurring at time t and at the final maturity T, but not in between.

Definition 2.1. For any fixed $t \in [0,T]$,

- the surface $\omega_t(\cdot,\cdot)$ is free of calendar spread arbitrage if $T \mapsto \omega_t(k,T)$ is increasing, for any $k \in \mathbb{R}$;
- a slice $k \mapsto \omega_t(k,T)$ is free of butterfly arbitrage if the corresponding density if non-negative.

A surface is free of static arbitrage if it is free of both calendar and butterfly arbitrages.

As hinted by its very name, this notion of arbitrage is static, in the sense that it only concerns the marginal distributions of the stock price between the two time stamps t and T, viewed at time t, but does not involve any dynamic behaviour in the running time t. It is equivalent to the impossibility of locking an arbitrage by trading in the options and the stock at time t and at the expiries of the options. Before discussing a dynamic version of no-arbitrage, let us recall the SVI parameterisation, which has become standard in industry practice on equity markets, and which constitutes the backbone of our analysis.

2.2. **SSVI parameterisation.** Finding a parametric surface free of static arbitrage has long been a challenge, and several suggestions were proposed in the literature. The real breakthrough came when Gatheral [14] disclosed the SVI parameterisation for the total variance as

$$\omega(k,T) = \text{SVI}(k) := a + b\left(\rho\,(k-m) + \sqrt{(k-m)^2+\sigma^2}\right). \tag{3}$$

Since we consider in this subsection a fixed time t, we drop the dependence thereof in the notation without confusion. Here a, b, ρ, m and σ are parameters. This parameterisation only provides a characterisation of slices, so that the parameters are in principle different for each maturity T. The fit to market data is fairly good, and we refer the reader to [9] for an efficient and robust dimension reduction calibration method. However, obtaining sufficient and necessary conditions on the parameters in order to ensure absence of static arbitrage in the sense of Definition 2.1 is far from trivial, and a quick `Google` search easily convinces anyone of the many (unsuccessful) attempts. In order to take into account the maturity dimension (and hence the whole volatility surface, still without dynamics), Gatheral and Jacquier [17] extended (3) to the Surface SVI (SSVI) parameterisation, defined as

$$\omega(k,\theta(T)) := \frac{\theta(T)}{2}\left(1 + \rho\varphi(\theta(T))k + \sqrt{(\varphi(\theta(T))k+\rho)^2 + \overline{\rho}^2}\right), \tag{4}$$

where $T \mapsto \theta(T)$ is a non-decreasing and strictly positive function representing the at-the-money total implied variance, $\rho \in (-1,1)$, $\overline{\rho} := \sqrt{1-\rho^2}$, and φ is a smooth function from $\mathbb{R}_+^*$ to $\mathbb{R}_+^*$. This formulation in fact enables one to find sufficient conditions to ensure absence of static arbitrage:

Proposition 2.2. *[Theorems 4.1 and 4.2 in* [17]*]*

- *There is no calendar spread if*

$$\begin{cases} \partial_t\theta(t) \geq 0, & \text{for all } t > 0, \\ 0 \leq \partial_\theta(\theta\varphi(\theta)) \leq \dfrac{1+\overline{\rho}}{\rho^2}\varphi(\theta), & \text{for all } \theta > 0; \end{cases}$$

- *There is no butterfly arbitrage if, for all $\theta > 0$,*

$$\theta\varphi(\theta) \leq \min\left(\frac{4}{1+|\rho|}, 2\sqrt{\frac{\theta}{1+|\rho|}}\right).$$

2.2.1. *Symmetric SSVI.* The uncorrelated cae $\rho = 0$ leads to a symmetric smile in log-moneyness, and the wording symmetric SSVI is usually used in this case in lieu of uncorrelated SSVI. Since we consider here a single smile, we will only be interested in the Butterfly arbitrage issue. The no-Butterfly arbitrage condition in Proposition 2.2 reads, for any $\rho \in (-1, 1)$,

$$\theta\varphi(\theta) < \frac{4}{1+|\rho|} \qquad \text{and} \qquad \theta\varphi(\theta)^2 \le \frac{4}{1+|\rho|};$$

hence, in the uncorrelated case $\rho = 0$, it can be simplified as

$$\theta\varphi(\theta) \le \min\left(4, 2\sqrt{\theta}\right), \qquad \text{for all } \theta > 0.$$

In the uncorrelated case, a slight improvement, as an explicit necessary and sufficient formulation, was provided in [17]. Define the function $\mathfrak{B} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mathfrak{B}(\theta) := A(\theta)\mathbb{1}_{\{\theta<4\}} + 16\,\mathbb{1}_{\{\theta\ge4\}}, \tag{5}$$

where, for any $\theta > 0$,

$$A(\theta) = \frac{16\theta\zeta_\theta\left(\zeta_\theta+1\right)}{8\left(\zeta_\theta-2\right)+\theta\zeta_\theta\left(\zeta_\theta-1\right)}, \qquad \text{with} \qquad \zeta_\theta := \frac{2}{1-\theta/4} + \sqrt{\left(\frac{2}{1-\theta/4}\right)^2 + \frac{2}{1-\theta/4}}.$$

Corollary 2.3. *If $\rho = 0$, there is no Butterfly arbitrage if and only if $(\theta\varphi(\theta))^2 \le \mathfrak{B}(\theta)$ for all $\theta > 0$.*

It is easy to see that $\lim_{\theta\downarrow0}\sqrt{A(\theta)/\theta} = \mathfrak{c}$, with $\mathfrak{c} \approx 4.45$, so that we have approximately a gain of a factor two with respect to the simplified sufficient condition. We note in passing that extended versions of SSVI have since been developed, in particular in [7, 18, 21], where again necessary conditions are provided to ensure absence of static arbitrage.

3. Dynamic arbitrage-free volatility surfaces

Static arbitrage is by now well understood and has contributed to providing valid examples to generate market options data and to design interpolators and extrapolators of options quotes. In the static setting above, no dynamics was set for any of the ingredients. We now extend this framework to a dynamic setting, whereby the stock price as well as the implied volatility are allowed to evolve. We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ on which all processes and Brownian motions are well defined. We shall consider that the stock price process is stochastic and adapted to $(\mathcal{F}_t)$. The main novelty of our approach, which is common for the moment with the early works by Carr and Sun [4] and Carr and Wu [5], is to impose some dynamics for the total implied variance $(\omega_t(k,T))_{t\in[0,T]}$. The maturity $T > 0$ is fixed throughout the paper, and hence the implied volatility at time t only makes sense for $t \in [0, T)$. We first introduce the concept of dynamic arbitrage without specifying any dynamics.

3.1. Dynamic arbitrage and consistent total variance models.

We introduce the dynamic framework with the following definition. One important notational detail from now on is that we now write $k_t := \log(K/S_t)$ instead of k for the log-moneyness, emphasising the importance of the running time t. We recall that the maturity $T > 0$ is fixed throughout the analysis.

Definition 3.1. A consistent total variance model is a couple $(S_t, \omega_t(k_t,T))_{t\in[0,T],K>0}$ such that, up to T,

- (i) the process S is a strictly positive $\mathbb{Q}$-martingale;
- (ii) for every $K > 0$, the process $\omega_\cdot(k_\cdot, T)$ has continuous paths and is strictly positive on $[0, T)$;
- (iii) for every $K > 0$, $\omega_t(k_t, T)$ converges to zero almost surely as t approaches T;
- (iv) for every $K > 0$, the process C defined by $C_t := S_t\mathrm{BS}\left(k_t, \sqrt{\omega_t(k_t,T)}\right)$ is a $\mathbb{Q}$-martingale.

We denote $\mathfrak{V}_T$ the set of all consistent total variance models, and we say that there is no dynamic arbitrage if $\mathfrak{V}_T \ne \emptyset$.

By Put-Call-Parity we can equivalently replace the last item above by the martingale property of the Put price process, directly through the Black-Scholes Put pricing function. The following useful remark relaxes Condition (iv) above from, replacing it effectively by a local martingale assumption:

Lemma 3.2. *Let $K > 0$ and assume that the process $(C_t)_{t\in[0,T]}$ defined by $C_t := S_t\mathrm{BS}\left(k_t, \sqrt{\omega_t(k_t,T)}\right)$ is a $\mathbb{Q}$-local martingale. Then if S is a martingale, so is C.*

Proof. Indeed the process P defined as $P_t := C_t - (S_t - K)$ is a local martingale which is positive and uniformly bounded by K, hence a martingale, and therefore $C = P + (S - K)$ is a martingale as well. □

At first glance, there is no link between the dynamics of each option contract (indexed by K), so that the option contract could evolve in an inconsistent manner even when starting from a static arbitrage-free configuration; this is actually not the case due to Definition 3.1(iii):

Lemma 3.3. *Absence of dynamic arbitrage implies absence of Butterfly arbitrage.*

Proof. We claim that the Call price is the conditional expectation of the payoff $(S_T - K)_+$. Consider a Put option with price P_t. In absence of dynamic arbitrage, Definition 3.1 implies that $P_t = \mathbb{E}_t^{\mathbb{Q}}[P_T]$ and $P_T = \lim_{t\uparrow T} \mathrm{P}^{\mathrm{BS}}(S_t, K, \omega_t(k_t, T)) = (K - S_T)_+$ almost surely. Since the payoff P_T is uniformly bounded by K, then dominated convergence implies that $P_t = \mathbb{E}_t[(K - S_T)_+]$. Since $C_t(K, T) = \mathrm{C}^{\mathrm{BS}}(S_t, K, \omega_t(k_t, T)) = S_t - K - \mathrm{P}^{\mathrm{BS}}(S_t, K, \omega_t(k_t, T)) = S_t - K - \mathbb{E}_t[(K - S_T)_+] = \mathbb{E}_t[(S_T - K)_+]$, the claim follows. □

Note that Definition 3.1(iii) is not present in [4], and it seems that nothing prevents inconsistent situations to occur. If Conditions (i) and (iv) in Definition 3.1 hold, each individual price being a martingale, no arbitrage can be exploited from trading individually in the options or stocks. Yet Butterfly arbitrage (even in its simple form of the non-monotonicity of the Call option price with respect to the strike) could occur and be exploited. As an example, consider two European Call options, with maturity T and respective strikes K_1 and K_2, with $0 < K_1 < K_2$, with dynamics given by

$$C_t(K_1, T) = C_0(K_1) \exp\left(\sigma_1 B_t^{(1)} - \frac{1}{2}\sigma_1^2 t\right) \qquad \text{and} \qquad C_t(K_2, T) = C_0(K_2) \exp\left(\sigma_2 B_t^{(2)} - \frac{1}{2}\sigma_2^2 t\right),$$

for two independent Brownian motions $B^{(1)}$ and $B^{(2)}$. For any $\varepsilon > 0$, the crossing time $\widetilde{T}_\varepsilon := \inf\{t : C_t(K_2) > C_t(K_1) + \varepsilon\}$ is strictly positive with strictly positive probability, and hence the prices will become inconsistent at $\widetilde{T}_\varepsilon$.

3.2. **The dynamic symmetric SSVI.** In order to push the analysis further, we now need to specify some dynamics. The stock price is assumed to satisfy

$$\frac{\mathrm{d}S_t}{S_t} = \sqrt{v_t}\,\mathrm{d}B_t^S, \tag{6}$$

starting without loss of generality from $S_0 = 1$, for some Brownian motion B^S. Here the process $(v_t)_{t\geq 0}$ is left unspecified, but regular enough (and non-negative) so that (6) admits a unique weak solution. Since we want the process S to be a true martingale, we impose the Novikov condition

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T v_t \mathrm{d}t\right\}\right] < \infty. \tag{7}$$

At this stage, the stock price and its instantaneous variance v may be correlated. We consider a dynamic version of the uncorrelated SSVI parameterisation (4), namely

$$\omega_t(k_t, \theta_t) = \frac{\theta_t}{2}\left(1 + \sqrt{1 + \varphi_t^2 k_t^2}\right). \tag{8}$$

where, similar to (4), θ_t accounts now for the at-the-money total implied variance at time t corresponding to an option maturing at T. In the setting (8), we implicitly disentangled the link between the function φ_t and the curve $T \mapsto \theta_t$, by introducing the process notation (φ_t). We shall keep this terminology from now on, and note that reverting back to the classical SSVI form (4), for any fixed $t \in [0, T]$ boils down to a simple change of variables. We further assume that (θ_t) and (φ_t) are diffusion processes given by

$$\begin{cases} \mathrm{d}\theta_t &= \theta_{1,t}\mathrm{d}t + \theta_{2,t}\mathrm{d}B_t^\theta, \quad \theta_0 > 0, \\ \mathrm{d}\varphi_t &= \varphi_{1,t}\mathrm{d}t + \varphi_{2,t}\mathrm{d}B_t^\varphi, \quad \varphi_0 > 0, \\ \mathrm{d}\langle B^\theta, B^\varphi\rangle_t &= \varrho\,\mathrm{d}t, \end{cases} \tag{9}$$

where B^θ and B^φ are two Brownian motions. The (time-dependent) coefficients θ_1, θ_2 and φ_1, φ_2 are left unspecified, but may be taken as stochastic as long as they are adapted to the filtration F_t and such that the two stochastic diffusions admit unique weak solutions.

The purpose of this article is to investigate the existence of consistent total variance model of the form (6)-(8)-(9). Moreover, given the symmetry of the implied volatility (8) expressed in log-moneyness, we expect θ and φ to depend solely on the driving Brownian motion of v, which is assumed to be independent from B^S. The maturity T does not come into play explicitly in our parameterisation, but is present through θ_t and φ_t, meaning that we should write θ_t^T and φ_t^T, even if we do not do so for notational convenience.

Consider the change of measure $\mathrm{d}\mathbb{Q} = S_T \mathrm{d}\mathbb{P}$ induced by the martingale S (assuming $S_0 = 1$). Girsanov's theorem therefore implies that the process $\widetilde{W}$ defined by $\mathrm{d}\widetilde{W}_t := \mathrm{d}B_t^S - \sqrt{v_t}\mathrm{d}t$ is a $\mathbb{Q}$-Brownian motion. Hence a Call option $C_\cdot(K,T)$ is a $\mathbb{P}$-martingale if and only if the process $\widetilde{C}_\cdot(K,T)$, defined as $\widetilde{C}_t(K,T) := C_t(K,T)/S_t = \mathrm{BS}(k_t, \sqrt{\omega_t(k_t,T)})$, is a $\mathbb{Q}$-martingale. Obviously, the martingale property should hold for any given (K,T). The terminal condition on the Call prices under $\mathbb{P}$ is that $S_t\mathrm{BS}(k_t, \sqrt{\omega_t(k_t,T)})$ converges to the intrinsic payoff $(S_T - K)_+$ almost surely as t tends to T, which is granted as soon as $\omega_t(k_t,T)$ converges to zero almost surely. The objective is to find conditions on the parameters $\theta_{1,t}$, $\theta_{2,t}$, $\varphi_{1,t}$, $\varphi_{2,t}$ and ϱ in (9) ensuring no dynamic arbitrage.

Theorem 3.4. *If there is a consistent total variance model, in the sense of Definition 3.1, then necessarily,*

$$\begin{cases} \mathrm{d}\theta_t &= \dfrac{(\theta_t\varphi_t - 4)(\theta_t\varphi_t + 4)}{16} v_t \mathrm{d}t - \theta_t\varphi_t\sqrt{v_t}\,\mathrm{d}B_t, \\ \mathrm{d}\varphi_t &= \left(16 + 16\varphi_t^2\theta_t - \theta_t^2\varphi_t^2\right)\dfrac{\varphi_t v_t}{16\theta_t}\mathrm{d}t + \varphi_t^2\sqrt{v_t}\,\mathrm{d}B_t, \end{cases} \tag{10}$$

where B is a Brownian motion independent from B^S.

We stress that the statement is only necessary. Nothing grants the existence of an actual solution; moreover, even if one exists, the following three conditions should be checked for the solution to to be valid:

- both processes θ and φ should be positive almost surely;
- the no-arbitrage Condition 2.3 should hold;
- the boundary condition θ_T should be null almost surely;

Lemma 3.7 below in fact shows that existence is a real issue . An important remark here is that the Brownian motion B may not be related to the dynamics of the variance process (v_t), as the latter does not come into play at any stage in the computations. We will discuss this more in detail in Section 4 below.

Proof. To simplify the computations, introduce the notations

$$Y_t := \varphi_t k_t, \qquad \eta_t := h(\theta_t), \qquad \gamma_t := f(Y_t), \qquad \Omega_t := \gamma_t\eta_t,$$

with $f(y) := \sqrt{1+\sqrt{1+y^2}}$ and $h(\theta) := \sqrt{\theta/2}$. This in particular implies that, in the uncorrelated dynamic SSVI framework (8), the Call price function (1) simplifies to

$$C_t^{\mathrm{obs}}(K,T) = S_t\mathrm{BS}(k_t,\Omega_t) =: S_t\widetilde{C}_t(k_t,\Omega_t).$$

Itô's formula implies that, for any (fixed) $k \in \mathbb{R}$ and any $t > 0$, we can write

$$\mathrm{d}\widetilde{C}_t = \partial_1\mathrm{BS}(k_t,\Omega_t)\mathrm{d}k_t + \partial_2\mathrm{BS}(k_t,\Omega_t)\mathrm{d}\Omega_t + \frac{1}{2}\partial_{11}^2\mathrm{BS}(k_t,\Omega_t)\mathrm{d}\langle k\rangle_t + \frac{1}{2}\partial_{22}^2\mathrm{BS}(k_t,\Omega_t)\mathrm{d}\langle\Omega\rangle_t + \partial_{12}^2\mathrm{BS}(k_t,\Omega_t)\mathrm{d}\langle k,\Omega\rangle_t.$$

The derivatives of the $\mathrm{BS}(\cdot,\cdot)$ function are classical and straightforward:

$$\begin{array}{llll} \partial_1\mathrm{BS}(u,v) &= -\mathrm{e}^u\mathcal{N}(d_-), & \partial_2\mathrm{BS}(u,v) &= n(d_+), \\ \partial_{11}^2\mathrm{BS}(u,v) &= -\mathrm{e}^u\mathcal{N}(d_-) + \dfrac{n(d_+)}{v}, & \partial_{12}^2\mathrm{BS}(u,v) &= \left(\dfrac{1}{2} - \dfrac{u}{v^2}\right)n(d_+), \\ \partial_{22}^2\mathrm{BS}(u,v) &= \left(\dfrac{u^2}{v^3} - \dfrac{v}{4}\right)n(d_+). & & \end{array}$$

Now, we can write the dynamics for all the processes appearing in this equation as

$$
\begin{aligned}
\mathrm{d}k_t &= \sqrt{v_t}\mathrm{d}\widetilde{W}_t - \frac{v_t}{2}\mathrm{d}t & \mathrm{d}\langle k\rangle_t &= v_t\mathrm{d}t,\\
\mathrm{d}Y_t &= \varphi_t\mathrm{d}k_t + \frac{Y_t}{\varphi_t}\mathrm{d}\varphi_t + \mathrm{d}\langle\varphi,k\rangle_t, & \mathrm{d}\langle Y\rangle_t &= \varphi_t^2\mathrm{d}\langle k\rangle_t + \left(\frac{Y_t}{\varphi_t}\right)^2\mathrm{d}\langle\varphi\rangle_t + 2Y_t\mathrm{d}\langle\varphi,k\rangle_t,\\
\mathrm{d}\gamma_t &= f'(Y_t)\mathrm{d}Y_t + \frac{1}{2}f''(Y_t)\mathrm{d}\langle Y\rangle_t, & \mathrm{d}\langle\gamma\rangle_t &= f'(Y_t)^2\mathrm{d}\langle Y\rangle_t,\\
\mathrm{d}\Omega_t &= \eta_t\mathrm{d}\gamma_t + \gamma_t\mathrm{d}\eta_t + \mathrm{d}\langle\eta,\gamma\rangle_t, & \mathrm{d}\langle\Omega\rangle_t &= \eta_t^2\mathrm{d}\langle\gamma\rangle_t + \gamma_t^2\mathrm{d}\langle\eta\rangle_t + 2\eta_t\gamma_t\mathrm{d}\langle\eta,\gamma\rangle_t,\\
\mathrm{d}\eta_t &= h'(\theta_t)\mathrm{d}\theta_t + \frac{1}{2}h''(\theta_t)\mathrm{d}\langle\theta\rangle_t, & \mathrm{d}\langle\eta\rangle_t &= h'(\theta_t)^2\mathrm{d}\langle\theta\rangle_t,\\
\mathrm{d}\langle k,\Omega\rangle_t &= \eta_t f'(Y_t)\left(\varphi_t\mathrm{d}\langle k\rangle_t + \frac{Y_t}{\varphi_t}\mathrm{d}\langle k,\varphi\rangle_t\right) + \gamma_t\mathrm{d}\langle k,\eta\rangle_t, & \mathrm{d}\langle k,\eta\rangle_t &= h'(\theta_t)\mathrm{d}\langle k,\theta\rangle_t,\\
\mathrm{d}\langle\eta,\gamma\rangle_t &= h'(\theta_t)f'(Y_t)\left(\varphi_t\mathrm{d}\langle\theta,k\rangle_t + \frac{Y_t}{\varphi_t}\mathrm{d}\langle\theta,\varphi\rangle_t\right).
\end{aligned}
$$

In order for the option price to be a martingale, the drift part $\mathrm{d}\widetilde{C}$ has to be equal to zero. Assume now, as in the proposition, that $\langle\theta,\varphi\rangle_t = \psi_t = \langle\theta,k\rangle_t = 0$. Long and tedious computations (that we perform with the counter checks of Sympy) show that the latter can be written as $Z_t^2(1+Z_t)^2\mathcal{P}(Z_t)\mathrm{d}t$, where $\mathcal{P}(\cdot)$ is a fifth-order polynomial, where $Z_t := \sqrt{1+Y_t^2}$. In particular, each coefficient $\mathcal{P}_i$ of order $i = 0,\dots,5$ read

$$
\begin{aligned}
\mathcal{P}_5 &= (\mathfrak{p}_t - 16)\left(\varphi_t^2\theta_{2,t}^2 + \theta_t^2\varphi_{2,t}^2 + 2\sqrt{\mathfrak{p}_t}\chi_t\right),\\
\mathcal{P}_4 &= -2\theta_t\Big\{\left(8\varphi_t\sqrt{\mathfrak{p}_t} - \mathfrak{p}_t - 16\right)\varphi_t\chi_t - \varphi_t^4\theta_t\theta_{2,t}^2 + 16\varphi_t^4\theta_t\theta_{1,t} - 4\varphi_t^4\theta_{2,t}^2 + 16\varphi_t\mathfrak{p}_t\varphi_{1,t} - 4\mathfrak{p}_t\varphi_{2,t}^2 - 16\theta_t\varphi_{2,t}^2\Big\},\\
\mathcal{P}_3 &= \varphi_t\Big\{2\left(16 - \mathfrak{p}_t\right)\theta_t\chi_t + \varphi_t\left[\mathfrak{p}_t\left(\mathfrak{p}_t - 16\right)v_t + \mathfrak{p}_t\theta_{2,t}^2 - 2\theta_t^4\varphi_{2,t}^2 - 32\mathfrak{p}_t\theta_{1,t} + 8\varphi_t\sqrt{\mathfrak{p}_t}\theta_{2,t}^2 - 8\theta_t^3\varphi_{2,t}^2 + 16\theta_{2,t}^2\right]\Big\},\\
\mathcal{P}_2 &= 2\theta_t\Big\{(8\varphi_t\sqrt{\mathfrak{p}_t} - \mathfrak{p}_t - 16)\varphi_t\chi_t + 4\theta_t\left(\varphi_t^6\theta_t v - 4\varphi_t^4 v + 4\varphi_t^3\theta_t\varphi_{1,t} - 3\varphi_t^2\theta_t\varphi_{2,t}^2 - 4\varphi_{2,t}^2\right)\Big\},\\
\mathcal{P}_1 &= -\theta_t^2\left(\mathfrak{p}_t + 8\varphi_t\sqrt{\mathfrak{p}_t} + 16\right)\left(\varphi_t^4 v_t - \varphi_{2,t}^2\right),\\
\mathcal{P}_0 &= -16\mathfrak{p}_t\theta_t\left(\varphi_t^4 v - \phi_{2,t}^2\right),
\end{aligned}
$$

where we introduced $\mathfrak{p}_t := \varphi_t^2\theta_t^2$, and write χ_t for the drift term of the covariation $\mathrm{d}\langle\theta,\varphi\rangle_t$.

Now, the key remark is that the dependence on the strike K and the running spot S_t in the drift of $\mathrm{d}\widetilde{C}$ is only through Z_t, as both θ_t and φ_t are independent thereof. The derivation above is valid for any fixed K, and thus for all $K > 0$. The only way the drift condition can be achieved is therefore that $\mathcal{P}(Z_t)$ is identically null, which holds if and only if $\mathcal{P}_i = 0$ for each $i = 0,\dots,5$. This system, with unknown $(\theta_{1,t},\theta_{2,t},\varphi_{1,t},\varphi_{2,t},\chi_t)$, turns out to be solvable as $\mathfrak{S}_+$ and $\mathfrak{S}_-$, where

$$
\mathfrak{S}_\pm = \begin{pmatrix}
\frac{v_t}{16}\left(\left[2\varrho^2 - 1\right]\varphi_t^2\theta_t^2 + 8\varphi_t^2\theta_t\left[\varrho^2 - 1\right] - 2\varphi_t^2\theta_t\left[\theta_t + 4\right]\overline{\varrho}|\varrho| - 16\right)\\
\pm\frac{\varphi_t\theta_t\sqrt{v_t}}{\varrho}\left(|\varrho|\overline{\varrho} - \varrho^2\right)\\
\frac{v_t\varphi_t}{16\theta_t}\left(8\varphi_t^2\theta_t\left[1+\varrho^2\right] - \varrho^2\varphi_t^2\theta_t^2 - 16\varrho^2 + \left[16 - 8\varphi_t^2\theta_t + \varphi_t^2\theta_t^2\right]\overline{\varrho}|\varrho| + 32\right)\\
\mp\varphi_t^2\sqrt{v_t}\\
\varphi_t^3\theta_t v_t\left(\overline{\varrho}|\varrho| - \varrho^2\right)
\end{pmatrix},
$$

where $\overline{\varrho} := \sqrt{1-\varrho^2}$. A quick look through the solution reveals the presence of the $\sqrt{\varrho^1 - 1}$, which only makes sense if $\varrho \in \{-1, 1\}$. In that case, the solution simplifies to the form stated in the theorem. Note that the two solutions for $\theta_{2,t}$ and $\varphi_{2,t}$ are symmetric in sign, so that they are equivalent, as Brownian increments are symmetric.

We now show that the decorrelation condition is in fact necessary. Denote the expression above $\mathrm{d}\widetilde{C}_+$, and introduce $\mathrm{d}\widetilde{C}_-$ as the same one, except that we now set $Y_t := -\sqrt{Z_t^2 - 1}$. Since the smile is symmetric by assumption, then clearly the difference $\mathrm{d}(\widetilde{C}_+ - \widetilde{C}_-)$ has to be null. This yields the following system:

$$
\begin{cases}
\theta_t^2\varphi_t^2\mathrm{d}\langle\varphi,k\rangle_t &= 0,\\
\left(\theta_t^2\varphi_t^2 + 8\theta_t\varphi_t^2 + 16\right)\theta_t\mathrm{d}\langle\varphi,k\rangle_t &= 0,\\
\varphi_t\Big(\mathrm{d}\langle\theta,k\rangle_t\theta_t^2\varphi_t^2 - 8\mathrm{d}\langle\theta,k\rangle_t\theta_t\varphi_t^2 + 16\mathrm{d}\langle\theta,k\rangle_t - 8\mathrm{d}\langle\varphi,k\rangle_t\theta_t^2\varphi_t\Big) &= 0,\\
\Big(\mathrm{d}\langle\theta,k\rangle_t\varphi_t + \mathrm{d}\langle\varphi,k\rangle_t\theta_t\Big)\Big(\theta_t^2\varphi_t^2 + 16\Big) &= 0,
\end{cases}
$$

and it is easy to see that the only valid solution is $\mathrm{d}\langle\theta,k\rangle_t = \mathrm{d}\langle\varphi,k\rangle_t = 0$. □

We now investigate whether the solution in Theorem 3.4 is trivial or not. Notice from (10) that Itô's product rule yields $\mathrm{d}(\theta_t\varphi_t) = 0$, so that there exists a constant ψ_T such that $\psi_T = \theta_t\varphi_t$. Hence, the SDE for $(\theta_t)_{t\in[0,T]}$ in (10) can be rewritten in a more compact way as

$$\mathrm{d}\theta_t = \frac{(\psi_T - 4)(\psi_T + 4)}{16} v_t\mathrm{d}t - \psi_T\sqrt{v_t}\,\mathrm{d}B_t, \qquad \text{with boundary condition } \theta_T = 0.$$

Proposition 3.5. *If $\mathfrak{V}_T \neq \emptyset$ and (θ,φ) solves (10), then for any $t \in [0,T]$, $\theta_t = \int_t^T v_u\mathrm{d}u$, all the smiles (at t, maturing at T) are flat, and $\int_t^T v_u\mathrm{d}u \in \mathcal{F}_t$.*

An immediate consequence is that the Black-Scholes model with deterministic volatility provides a consistent total variance model.

Proof. The condition $\theta_T = 0$ implies $\psi_T = 0$. The SSVI smile is then trivial, equal to θ_t for all strikes. Furthermore $\mathrm{d}\theta_t = -v_t\mathrm{d}t$, and hence $\theta_t = \int_t^T v_u\mathrm{d}u$. Since θ_t is $\mathcal{F}_t$ measurable, then so is $\int_t^T v_u\mathrm{d}u$. □

Remark 3.6. If the property $\int_t^T v_u\mathrm{d}u \in \mathcal{F}_t$ is taken to hold for every $T > 0$, then v_T is $\mathcal{F}_t$-measurable, and hence v is deterministic. Note that we only considered here a fixed maturity T, not all of them. One could then wonder whether the necessary properties above imply that v is deterministic, in which case the only solution would be a time-dependent Black-Scholes model. We provide a non-trivial example showing that this is not the case in general: on the filtration of a planar Brownian motion (W,B), fix $T = 1$ and $0 < \varepsilon_1, \varepsilon_2 < 1$, and define

$$v_t := \begin{cases} v_0, & \text{for } t \in [0,1/3),\\ v_0(1+\varepsilon_1)1\!\!1_{B_{1/3}>0} + v_0(1-\varepsilon_2)1\!\!1_{B_{1/3}\leq 0}, & \text{for } t \in [1/3,2/3),\\ v_0(1-\varepsilon_1)1\!\!1_{B_{1/3}>0} + v_0(1+\varepsilon_2)1\!\!1_{B_{1/3}\leq 0}, & \text{for } t \in [2/3,1). \end{cases}$$

Then clearly $\int_t^T v_u\mathrm{d}u$ is $\mathcal{F}_t$-measurable for every $t \in [0,1]$, and therefore the conditional law of S_T is lognormal, and the smiles are flat at every point in time. This in particular yields another example of a consistent total variance model, different from the time-dependent Black-Scholes.

Lemma 3.7. *Let $\alpha := -\left(\frac{\psi_T}{16} - \frac{1}{\psi_T}\right)$. If $\mathbb{E}\left[\exp\left\{2\alpha^2\int_0^T v_t\mathrm{d}t\right\}\right]$ is finite (in particular, under the Novikov condition (7) if $\alpha < \frac{1}{2}$), then any solution of (10) on $[0,T]$ satisfies*

$$\theta_t = \frac{\psi_T}{2\alpha}\log\mathbb{E}_t\left[\exp\left(\frac{2\alpha\theta_T}{\psi_T}\right)\right], \quad \textit{almost surely for all } t \in [0,T].$$

In particular, $\theta_T = 0$ almost surely entails that $\theta_t = 0$ almost surely for all $t \in [0,T]$.

Proof. Setting $\widetilde{\theta}_t := -\theta_t/\psi_T$ yields

$$\mathrm{d}\widetilde{\theta}_t = -\left(\frac{\psi_T}{16} - \frac{1}{\psi_T}\right)v_t\mathrm{d}t + \sqrt{v}_t\mathrm{d}B_t, \tag{11}$$

with $\widetilde{\theta}_T = 0$. Let $\alpha := -\left(\frac{\psi_T}{16} - \frac{1}{\psi_T}\right)$, so that $\mathrm{d}\widetilde{\theta}_t = \alpha v_t\mathrm{d}t + \sqrt{v}_t\mathrm{d}B_t$. Itô's formula on $[t,T]$ implies that

$$\begin{aligned} \mathrm{e}^{-2\alpha\widetilde{\theta}_T} &= \mathrm{e}^{-2\alpha\widetilde{\theta}_t} - 2\alpha\int_t^T \mathrm{e}^{-2\alpha\theta_u}\mathrm{d}\widetilde{\theta}_u + 2\alpha^2\int_t^T \mathrm{e}^{-2\alpha\widetilde{\theta}_u}v_u\mathrm{d}u\\ &= \mathrm{e}^{-2\alpha\widetilde{\theta}_t} - 2\alpha\int_t^T \mathrm{e}^{-2\alpha\widetilde{\theta}_u}\left[\alpha v_u\mathrm{d}u + \sqrt{v}_u\mathrm{d}B_u\right] + 2\alpha^2\int_t^T \mathrm{e}^{-2\alpha\widetilde{\theta}_u}v_u\mathrm{d}u\\ &= \mathrm{e}^{-2\alpha\widetilde{\theta}_t} - 2\alpha\int_t^T \mathrm{e}^{-2\alpha\widetilde{\theta}_u}\sqrt{v}_u\mathrm{d}B_u. \end{aligned}$$

Note that $-2\alpha\widetilde{\theta}_u > 0$ and $\mathrm{e}^{-2\alpha\widetilde{\theta}_u} = \mathrm{e}^{-2\alpha\widetilde{\theta}_0}\mathrm{e}^{-2\alpha^2\int_0^u v_t\mathrm{d}t}\mathrm{e}^{-2\alpha\int_0^u\sqrt{v}_t\mathrm{d}B_t}$; the Novikov condition applied to the process $-2\alpha\int_0^\cdot\sqrt{v}_t\mathrm{d}B_t$ reads $\mathbb{E}\left[\exp\left\{2\alpha^2\int_0^T v_t\mathrm{d}t\right\}\right] < \infty$, in which case the stochastic integral is square integrable with null expectation, and so that every term in the expression has finite expectation. Taking expectations conditional on $\mathcal{F}_t$ on both sides yields $\mathbb{E}_t\left[\mathrm{e}^{-2\alpha\widetilde{\theta}_T}\right] = \mathrm{e}^{-2\alpha\widetilde{\theta}_t}$, and

$$\widetilde{\theta}_t = -\frac{1}{2\alpha}\log\mathbb{E}_t\left[\mathrm{e}^{-2\alpha\widetilde{\theta}_T}\right], \quad \text{almost surely for all } t \in [0,T],$$

and hence

$$\theta_t = \frac{\psi_T}{2\alpha}\log\mathbb{E}_t\left[\exp\left(\frac{2\alpha\theta_T}{\psi_T}\right)\right], \quad \text{almost surely for all } t\in[0,T].$$

Imposing $\theta_T = 0$ almost surely implies that $\theta_t = 0$ almost surely for all $t \in [0,T]$. □

Remark 3.8. Because of the terminal condition $\theta_T = 0$, the stochastic differential equation satisfied by θ is actually a classical example of a BSDE. Consider indeed the process

$$X_t = \xi + \int_t^T f(s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}B_s, \tag{12}$$

on $[0,T]$, with terminal condition $X_T = \xi$, where ξ is a bounded, $\mathcal{F}_T^B$-measurable random variable, and $f : [0,T]\times\mathbb{R}\to\mathbb{R}$ has at most quadratic growth. This is the classical (one-dimensional) example of a quadratic BSDE [38, Chapter 10]. In our setting, integrating the SDE for $\widetilde{\theta}$ on $[t,T]$ reads

$$\widetilde{\theta}_t = \widetilde{\theta}_T + \int_t^T \left(\frac{\psi_T}{16} - \frac{1}{\psi_T}\right) v_s \mathrm{d}s - \int_t^T \sqrt{v}_s \mathrm{d}B_s,$$

which is exactly of the form (12) with $X = \widetilde{\theta}$ $f(\cdot, z) \equiv \left(\frac{\psi_T}{16} - \frac{1}{\psi_T}\right) z^2$, $Z = \sqrt{v}$ and $\xi = 0$ almost surely. From [38, Chapter 10], it is known that if (X,Z) is a solution to (12) with X bounded, then $Z \in \mathbb{H}^2_{\mathrm{BMO}}$, where the space $\mathbb{H}^2_{\mathrm{BMO}}$ is defined as

$$\mathbb{H}^2_{\mathrm{BMO}} := \left\{\varphi \in \mathbb{H}^2 : \left\|\int_0^\cdot \varphi_s \mathrm{d}B_s\right\|_{\mathrm{BMO}} < \infty\right\},$$

with $\mathbb{H}^2$ the space of square integrable martingales, and

$$\|M\|_{\mathrm{BMO}} := \sup_{\tau\in\mathcal{T}_0^T} \left\|\mathbb{E}\left[\langle M\rangle_T - \langle M\rangle_\tau | \mathcal{F}_\tau\right]\right\|_\infty,$$

where $\mathcal{T}_0^T$ denotes the set of stopping times in $[0,T]$. Full details and properties of BMO spaces can be found in [28]. Since the driver f is quadratic and smooth, uniqueness of a solution to (12) is guaranteed by [38, Theorem 10.5]. We shall only be interested in the case where the terminal condition ξ is bounded here. More general results exist, but are not needed in our setting. Existence of a unique solution to (12) is then given by [38, Theorem 10.6], with upper bound estimate for the (appropriate) norms of X and Z.

4. Smile bubbles

We showed previously that, except for dynamics close to Black-Scholes, there is not dynamic model for $(\theta_t)_{t\in[0,T]}$ and $(\varphi_t)_{t\in[0,T]}$ of symmetric SSVI up to the fixed maturity T. We now investigate what happens if we restrict ourselves to a shorter time horizon; more precisely, we wish to find a valid dynamic model for θ up to some (possibly stochastic) time horizon $\tau \in (0,T)$. This would therefore constitute a dynamic arbitrage-free model on $[0,\tau]$, satisfying the martingale condition for the stock and Vanilla option price, with total variance given by SSVI. One caveat is that absence of Butterfly arbitrage is not granted, because the Vanilla prices can a priori no longer be written as conditional expectations of the terminal payoff, although the individual option prices are martingales. Now, in the case of symmetric SSVI, we have explicit necessary and sufficient conditions on the parameters preventing Butterfly arbitrage: so starting in this domain, we will have locally no dynamic nor Butterfly arbitrages up to the exit time τ of this domain. In order to state the definitions below, let $\mathcal{T}_T$ denote the set of all stopping times with values in $[0,T]$ (with $T > 0$ still fixed). The following definition can be seen as a local version of Definition 3.1.

Definition 4.1. A locally consistent total variance model is a couple $(S_\cdot, \omega_\cdot(k_\cdot, T))$ such that there exists $\tau \in \mathcal{T}_T$ for which, up to τ:

(i) the process S is a strictly positive $\mathbb{Q}$-martingale;
(ii) for every $K > 0$, the process $\omega_\cdot(k_\cdot, T)$ has continuous paths and is strictly positive;
(iii) there is no Butterfly arbitrage.
(iv) for every $K > 0$, the process C defined by $C_t := S_t\mathrm{BS}\left(k_t, \sqrt{\omega_t(k_t,T)}\right)$ is a $\mathbb{Q}$-martingale.

We denote $\mathfrak{V}_T^{\mathrm{loc}}$ the set of all locally consistent total variance models, and we say that there is no dynamic arbitrage on $[0,\tau]$ if $\mathfrak{V}_T^{\mathrm{loc}}$ is not empty.

Remark 4.2. Here again we assume that the implied volatility of Calls and Puts is the same, so that the Put-Call-Parity holds by assumption, and we can equivalently replace Definition 4.1(iv) by the martingale property of the Put price process, and Definition 4.1(iii) by the corresponding properties of the Put price. Also the martingale property of the option prices is equivalent here again to the local martingale property.

Obviously, $\mathfrak{V}_T \subset \mathfrak{V}_T^{\mathrm{loc}}$, i.e. every consistent total variance model can be localized. In order to distinguish between the two definitions, we introduce the concept of a smile bubble as the existence of a strict locally consistent variance model:

Definition 4.3. A smile bubble is an element of $\mathfrak{V}_T^{\mathrm{loc}}$ which can not be extended to an element of $\mathfrak{V}_T$.

Note that a smile bubble cannot last until the expiry T, by definition, whence the term 'bubble'. If one trades only within a bubble lifespan, namely on $[0, \tau]$, with possibly additionally unwinding trades at the expiry T, there is no arbitrage to be made. Arbitrage will follow from purely dynamic strategies involving unwinding positions beyond the bubble lifespan. We believe this set-up, albeit not very surprising from a strict mathematical point of view, might be very relevant to account for real life joint underlying and options dynamics. Indeed, if one goes back to the local dynamic of an SSVI bubble found in Theorem 3.4, we stress the fact that the driving Brownian motion of the SSVI parameters should be independent of the driving Brownian motion of S, with no necessary relation to the Brownian motion (if any) driving the instantaneous volatility process v. This might correspond to the empirical finding in the study of the high frequency joint dynamics of the instantaneous volatility and of option prices on equity index options in [1], where the noise driving the non-Delta move of the option prices does not correspond to the idiosyncratic noise of the instantaneous volatility. In fact, beyond this high-frequency situation, it might be the case that the joint dynamic of the underlying of an option and of the Vanilla option prices is a succession of bubbles, with some bubbles far from a fully consistent joint dynamic, and other ones closer.

4.1. **Symmetric SSVI smile bubbles.** In the symmetric SSVI dynamics, we just proved that there cannot exist any consistent total variance model. We did so by identifying a local dynamics that (θ, φ) should satisfy in order for option prices to be local martingales. The only ingredient missing to design a bubble is the no-Butterfly arbitrage condition; in the case of symmetric SSVI, we have an explicit description of the no-Butterfly domain in terms of the parameters, as discussed above. So assuming the model parameters start within this domain, and making them evolve according to the local dynamics identified in our computations, we indeed obtain a bubble. The bubble lifespan is then the first (stopping) time when either θ or φ becomes negative, or when we exit the no-Butterfly domain. The explicit description of the no-Butterfly arbitrage region is given by Condition 2.3, so that combining it with (10), we have $\psi_T^2 \leq \mathfrak{B}(\theta_t)$ for all $t \in [0, T]$. The function $\mathfrak{B}$ is smooth and increasing from $[0, \infty)$ to $[0, 16]$, so that its inverse $\mathfrak{B}^{\leftarrow}$ is well defined from $[0, 16]$ to $[0, 4]$, and $\theta_0 \geq \mathfrak{B}^{\leftarrow}(\psi_T^2)$, implying in particular that θ is non-negative. Therefore, if we assume $\psi_T < 4$, then $\theta_0 > \mathfrak{B}^{\leftarrow}(\psi_T^2)$, and we can consider the stopping time

$$\tau := \inf\left\{t \leq T, \theta_t < \mathfrak{B}^{\leftarrow}(\psi_T^2)\right\} \in \mathcal{T}_T. \tag{13}$$

On $[0, \tau]$ both Vanilla option prices and stock price are martingales, and the no-Butterfly condition holds, so that we have an explicit description of the symmetric SSVI implied volatility bubbles. These however depend on the parameter ψ_T and on the dynamics of the process v, which partially drives θ. Indeed, from (10),

$$\theta_t = \theta_0 + \left(\frac{\psi_T^2}{16} - 1\right) \int_0^t v_u \mathrm{d}u + \psi_T \int_0^t \sqrt{v_u}\, \mathrm{d}B_u^{\theta},$$

where B^θ is a Brownian motion independent from B^S, which may, or may not, be related to the Brownian motion driving v. We need to check that θ remains positive, but assuming that θ_0 satisfies the no-Butterfly constraint in Condition 2.3, the positivity of θ before time τ follows from its definition. Therefore any locally integrable instantaneous process independent from B^S defines a symmetric SSVI implied volatility bubble up to the stopping time τ. Note in passing that the Novikov condition on v grants the martingale property of S, but could certainly be weakened in this bubble context.

Due to the finite lifespan of the bubble, shorter than the options' maturity, option prices are not given by the expectation of the terminal payoff under the risk-neutral dynamic of the underlying. It is natural in the context of a bubble to then take the conditional expectation at the bubble time boundary. Since option prices are true martingales, Doob's optional stopping theorem [33, Chapter II, Section 3], implies that the price of the option at time $t \le \tau$ is $\mathbb{E}_t\left[S_\tau \mathrm{BS}(k_\tau, \varpi_\tau)\right]$, where $S_. = K\exp(-k_.)$, with the notation

$$\varpi_t := \frac{1}{2}\left(\theta_t + \sqrt{\theta_t^2 + \psi_T^2 k_t^2}\right)$$

for clarity in the forthcoming computations. Note that, for any $t \in [0,T]$, ϖ_t depends on the terminal expiry T through ψ_t. Furthermore, the option price is also given by the Black-Scholes formula composed with SSVI, so that (recall that $S_0 = 1$, hence $k_0 = \log K$)

$$S_0 \mathrm{BS}(k_0, \varpi_0) = \mathbb{E}_0\left[S_\tau \mathrm{BS}(k_\tau, \varpi_\tau)\right]. \tag{14}$$

This equality holds irrespective of the dynamics of the variance process v. Of course the latter will directly impact the joint law of $(\tau, k_\tau, \theta_\tau)$. Since the strike can be chosen arbitrarily, this in turn yields an interesting property of this joint law, of which we provide several examples below. We will christen (14) the *Bubble Master Equation* in the sequel. It will be convenient in some situations to go back to a Brownian setting by time-change: let us restart from

$$\mathrm{BS}(k_0, \varpi_0) = \mathbb{E}_0\left[\mathbb{1}_{\{\tau<T\}}\mathrm{BS}(k_\tau, \varpi_\tau) + \mathbb{1}_{\{\tau\ge T\}}\mathrm{BS}(k_T, \varpi_T)\right].$$

By time change, there exist two Brownian motions β^θ, β^S, independent conditionally on $(v_t)_{t\ge0}$, such that

$$\theta_t = \theta_0 + \left(\frac{\psi_T^2}{16} - 1\right)t + \psi_T \beta_t^\theta, \qquad S_t = S_0 \exp\left\{\beta_t^S - \frac{t}{2}\right\}, \quad \text{and} \quad \tau = \inf\left\{t < \mathcal{V}, \theta_r < \mathfrak{B}^{\leftarrow}(\psi_T^2)\right\}, \tag{15}$$

where $\mathcal{V}_T := \int_0^T v_s \mathrm{d}s$. Introduce now the new probability measure $\mathbb{Q}$ via $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_T} := \mathcal{E}(L_T)$, where $L_T := \eta\beta_T^\theta$ and $\eta = -\frac{\psi_T}{16} + \frac{1}{\psi_T} > 0$, so that the process $\widetilde{\beta}$ defined by $\widetilde{\beta}_t = -\beta_t^\theta + \eta t$ is then a $\mathbb{Q}$-Brownian motion by Girsanov's Theorem. Furthermore, τ is now the first hitting time of $\widetilde{\beta}$ of the level $a := \frac{1}{\psi_T}\left(\theta_0 - \mathfrak{B}^{\leftarrow}(\psi_T^2)\right)$, and hence, by conditioning,

$$\mathrm{BS}(k_0, \varpi_0) = \mathbb{E}_0^{\mathbb{Q}}\left[\exp\left\{\eta\widetilde{\beta}_\tau - \frac{\eta^2\tau}{2}\right\}\mathbb{1}_{\{\tau<\mathcal{V}_T\}}\mathrm{BS}(k_\tau, \varpi_\tau) + \exp\left\{\eta\widetilde{\beta}_{\mathcal{V}_T} - \frac{\eta^2\mathcal{V}_T}{2}\right\}\mathbb{1}_{\{\tau\ge\mathcal{V}_T\}}\mathrm{BS}(k_{\mathcal{V}_T}, \varpi_{\mathcal{V}_T})\right]. \tag{16}$$

4.1.1. *The Black-Scholes-SSVI bubble.* We will investigate the Bubble master equation when T goes to infinity, choosing ψ_T such that it is constantly equal to a given $\psi_\infty < 4$. Assume $v_t = v_0 > 0$ almost surely for all $t \in [0,T]$; then $\mathcal{V}_T = vT$ diverges to infinity as T increases. In this limiting case, we deduce from (16):

Proposition 4.4. *For any $\psi_\infty < 4, \theta_0 > \mathfrak{B}^{\leftarrow}(\psi_\infty^2)$, $k_0 \in \mathbb{R}$, with $\boldsymbol{\beta} := \mathfrak{B}^{\leftarrow}(\psi_\infty^2)$, we have*

$$\mathrm{BS}(k_0, \varpi_0) = \frac{a\mathrm{e}^{\eta a}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\frac{y}{2}}\mathrm{BS}\left(k-y, \frac{1}{2}\left(\boldsymbol{\beta} + \sqrt{\psi_\infty^2(k-y)^2 + \boldsymbol{\beta}^2}\right)\right)\sqrt{\frac{4\eta^2+1}{y^2+a^2}}\mathrm{K}_1\left(\frac{\sqrt{(4\eta^2+1)(y^2+a^2)}}{2}\right)\mathrm{d}y,$$

where $\eta := \frac{1}{\psi_\infty} - \frac{\psi_\infty}{16}$ and $a := \frac{\theta_0 - \boldsymbol{\beta}}{\psi_\infty} > 0$, $\mathrm{K}_1(\cdot)$ denoting the modified Bessel function of the second kind.

Proof. Since $a, \eta > 0$, then $\eta\widetilde{\beta}_T < \eta a$, so that the second term in (16) tends to zero pointwise as T tends to infinity, while remaining bounded above by $K\mathrm{e}^{\eta a}$. Furthermore, the first term in (16) increases to

$$\exp\left\{\eta\widetilde{\beta}_\tau - \frac{\eta^2\tau}{2}\right\}\mathrm{BS}(k_\tau, \varpi_\tau)\mathbb{1}_{\{\tau<T\}},$$

since τ is finite almost surely. So, by dominated convergence for the second term and monotone convergence for the first one, we obtain

$$\mathrm{BS}(k_0, \varpi_0) = \mathbb{E}_0^{\mathbb{Q}}\left[\exp\left\{\eta\widetilde{\beta}_\tau - \frac{\eta^2\tau}{2}\right\}\mathrm{BS}(k_\tau, \varpi_\tau)\right].$$

Under $\mathbb{Q}$, using (15), we have $k_t = \log\left(\frac{K}{S_t}\right) = k - \beta_t^S + \frac{1}{2}$, so that

$$\mathrm{BS}(k_0, \varpi_0) = \mathbb{E}_0^{\mathbb{Q}}\left[\mathrm{e}^{\eta a - \frac{1}{2}\eta^2\tau}\mathrm{BS}\left(k - \beta_\tau^S + \frac{\tau}{2}, \frac{1}{2}\left\{\boldsymbol{\beta} + \sqrt{\psi_\infty^2\left(k - \beta_\tau^S + \frac{\tau}{2}\right)^2 + \boldsymbol{\beta}^2}\right\}\right)\right].$$

Under $\mathbb{Q}$, the density of τ is given by $p(t) := \frac{a}{\sqrt{2\pi t^3}} \exp\left\{-\frac{a^2}{2t}\right\}$. Given the independence of τ and β^S, the right-hand side then reads

$$\mathrm{e}^{\eta a}\int_0^\infty \left[\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\eta^2 t}\mathrm{BS}\left(k - x\sqrt{t} + \frac{t}{2}, \frac{1}{2}\left\{\boldsymbol{\beta} + \sqrt{\psi_\infty^2\left(k - x\sqrt{t} + \frac{t}{2}\right)^2 + \boldsymbol{\beta}^2}\right\}\right) p(t)\frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x\right]\mathrm{d}t$$

$$= a\frac{\mathrm{e}^{\eta a}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\frac{y}{2}}\mathrm{BS}\left(k - y, \frac{1}{2}\left(\boldsymbol{\beta} + \sqrt{\psi_\infty^2 (k-y)^2 + \boldsymbol{\beta}^2}\right)\right)\left(\int_0^\infty \mathrm{e}^{-\frac{1}{2}\left(\eta^2 + \frac{1}{4}\right)t}\mathrm{e}^{-\frac{y^2+a^2}{2t}}\frac{\mathrm{d}t}{t^2}\right)\mathrm{d}y,$$

with $y := x\sqrt{t} - \frac{1}{2}t$. The proposition follows from the computation of the inner integral as

$$\int_0^\infty \mathrm{e}^{-\frac{1}{2}\left(\eta^2 + \frac{1}{4}\right)t}\mathrm{e}^{-\frac{y^2+a^2}{2t}}\frac{\mathrm{d}t}{t^2} = \sqrt{\frac{4\eta^2+1}{y^2+a^2}}\mathrm{K}_1\left(\frac{\sqrt{(4\eta^2+1)(y^2+a^2)}}{2}\right).$$

□

Remark 4.5. It can be noticed that in fact, we can apply exactly the same reasoning for any positive level β smaller than θ_0 since we use only the martingale property of the price under the dynamic of θ: the same identity holds therefore for any $0 < \beta < \theta_0$ not only for the point $\mathfrak{B}^{\leftarrow}(\psi_\infty^2)$. Observe that the right-hand side does not depend on $\boldsymbol{\beta}$, since the left-hand side does not. Interestingly enough, this equality does not seem easy to obtain by classical means.

4.1.2. *The Heston-SSVI bubble.* We now investigate the existence of an SSVI bubble in the Heston model, that is when the variance process $(v_t)_{t\geq 0}$ is a Feller diffusion of the form

$$\mathrm{d}v_t = \kappa(\overline{v} - v_t)\mathrm{d}t + \xi\sqrt{v_t}\,\mathrm{d}B_t^v, \qquad v_0 > 0,$$

with $\kappa, \overline{v}, \xi > 0$. The Yamada-Watanabe conditions [27, Proposition 2.13, page 291] ensure that a unique strong solution exists, and the Feller condition $2\kappa\overline{v} \geq \xi^2$ that the latter never reaches the origin. We consider two cases, depending on whether the Brownian motions B^v and B^θ are fully or anti correlated (and always independent of B^S). If they are fully correlated, then the SDE (10) satisfied by θ reads

$$\mathrm{d}\theta_t = \left(\frac{\psi_T^2}{16} - 1\right)v_t\mathrm{d}t - \psi_T\sqrt{v_t}B_t^\theta = \left(\frac{\psi_T^2}{16} - 1\right)v_t\mathrm{d}t - \frac{\psi_T}{\xi}\left(\mathrm{d}v_t - \kappa(\overline{v} - v_t)\mathrm{d}t\right),$$

so that

$$\theta_t = \theta_0 + \left(\frac{\psi_T^2}{16} - 1 - \frac{\kappa\psi_T}{\xi}\right)\mathcal{V}_t - \frac{\psi_T}{\xi}(v_t - v_0) + \frac{\kappa\overline{v}\psi_T t}{\xi}.$$

In the case where $\langle B^\theta, B^v\rangle_t = -\mathrm{d}t$, then the computations above become

$$\theta_t = \theta_0 + \left(\frac{\psi_T^2}{16} - 1 + \frac{\kappa\psi_T}{\xi}\right)\mathcal{V}_t + \frac{\psi_T}{\xi}(v_t - v_0) - \frac{\kappa\overline{v}\psi_T t}{\xi}.$$

We can then proceed as in the Black-Scholes case, at least for the innermost conditional expectation; the subsequent computations are still more intricate, since in general the law of τ is unknown. The symmetric SSVI bubble equation provides in this case some information of the joint law of τ, v_τ and $\mathcal{V}_\tau$.

5. Conclusion

We have investigated the possibility of consistent joint dynamics of a stock price and SSVI smiles, at least in the case of symmetric smiles, given a finite time horizon. We proved that this problem is in fact infeasible, albeit with two fundamental by-products: first, a generalisation to non-symmetric smiles could potentially provide a non-trivial answer, but the length of the computations would certainly benefit from a more machine-enhanced approach; second, we identified a new regime of implied volatility bubbles, which by definition vanish at maturity. These bubbles are singular, however, as arbitrage opportunities (with strategies including unwinding positions at maturity), despite present, cannot be achieved during the lifespan of the bubble. These bubbles might account for the empirical findings in the high-frequency joint dynamics of an Equity index and its options, as in [1]. One could even conjecture that those bubbles are in fact more a rule than an exception, and that real-life dynamics over the lifespan of an option is a succession of such bubbles. We fully described the SSVI bubbles in the symmetric SSVI case.

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